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Time-Related Procedure to Structural Column Failure

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Based on a thought experiment, a slender column is loaded to the Euler column buckling load by bringing the platen of a testing machine down at a uniform rate. The shortening of the column y is proportional to the time t. Bending, that is, bowing out of the column, starts as soon as the compressive load $P > 0$. Every column to be tested has an initial accidental eccentricity, either because it is not perfectly straight or the mean resistance internally of the random arrangement of the crystalline structure of the grains of structural steel (or other metal) does not coincide with the central column axis. For a solid rectangular cross-section, the eccentricity e falls within a certain interval. The value of e is always greater than zero, but also has an upper bound so that the maximum compressive stress due to compression plus bending does not exceed the proportional limit σ_{pl}. Euler's formula for critical buckling P_{cr} when divided by the cross-sectional column area A does not yield a meaningful average stress. Only the maximum compressive fiber stress, at the midlength of the column is significant as it approaches σ_{pl}. The coordinates of the bowed column are solved for at any time t at the column midlength, until the column fails at time t_{cr}. It is also demonstrated that $P = f(y)$ is not a straight line, although it may come close. Adopting a convenient rate of compression of the column C_1 the time to failure t_{cr} can also be predicted for the corresponding, computed y_{cr}.

Introduction

TYPICALLY, work on the buckling of slender columns had started with the Euler equation, published in 1759, derived from a simple second-order homogeneous differential equation, namely

$$P_{cr} = \pi^2 EI/L^2 \tag{1}$$

where E is the modulus of elasticity, I the moment of inertia, and L is the length of the pin-ended column. P_{cr} is the critical compressive load that is the maximum the column can sustain when buckling is impending. Bending, that is, bowing out of the column starts as soon as the compressive load $P > 0$. Every column to be tested has an initial eccentricity, either because it is not perfectly straight or the mean resistance internally of the random arrangement of the crystalline structure of the grains of structural steel does not coincide with the central column axis and central loading.

Bleich [1] in Fig. 15, plotted critical average stress P_{cr}/A vs L/r, the slenderness ratio, for $0 \leq e/r \leq 2.0$, where r is the radius of gyration. These are considered for intentional eccentricities, computed for structural steel columns having rectangular cross sections. These graphs were based on theories and tests developed by von Karman and Chawalla, and simplified by Westergaard and Osgood. Bleich also presents a brief history of slender column buckling work going back to Euler, Engesser, Considere, Lagrange, Lamarle, and many other investigators.

Timosheriko [2] also plotted graphs of σ_c, so-called average compressive stress producing yielding, vs L/r for $0 \leq e/s \leq 0.9$ where e is a given intensional eccentricity and s is the section modulus. In the writer's paper e is an accidental eccentricity for values

$$0.00171 < e/s \leq 5.820$$

In most books at that time, they discuss δ, the maximum deflection or bowing out of a column, just before P_{cr} is reached, as indeterminate. In this paper, for a range of accidental eccentricities e the shortening of the column y is computed as related to the load P and, at buckling P_{cr} is related to y_{cr} (or t_{cr}), that is, the shortening of the bar is proportional to time if compressed at a uniform rate.

In the typical Euler derivation, Borg [3] has assumed some arbitrary deflection δ that a column can sustain as P approaches P_{cr}. The column buckles out, or fails, at or right after P_{cr}, but δ (in this paper called v) goes to zero if the load $P < P_{cr}$ is removed and there is no eccentricity. Thus, Euler, in his solution of a simple second-order differential equation, found that δ disappeared from his derived equation for P_{cr}, for a slender pin-ended column. Consider a slender column as L/r from about 120 to 300.

Wood [4] made a survey of changes in crystalline structure of polycrystalline metals and large single crystals subjected to deformation by cold working and by static and cyclic stressing, using x ray and other tests. It is shown that it is possible to observe internal lattice strains, which is related to the process of strain hardening. The deformation of a ductile metal raises the problems of the process by which the grains take up change in shape and hardening. The writer feels that such work as Wood's may lead to some numerical mean resultant of the resisting crystalline grains, thus leading to a better handle on accidental eccentricities.

Wright [5] plots a function of accidental eccentricity ratio η due to any cause, where a strength parameter verses some stress ratio, and $0 \leq \eta \leq 5$. So, as e increases, v, the bowing out decreases as P_{cr} is reached, so as not to exceed the proportional limit.

Reference [6]† states that there is always some eccentricity in the applied loading of a member due to initial imperfections. When a compressive load is increased, eccentricity sets up bending in the member causing it to deflect. This progresses to where the bending increases at a greater rate than the compressive loading and the member becomes unstable.

Bomel [7] states that structural analysis may be defined as the art of formulating a mathematical model so that one can peek: What is the probability that a structure behaves in a specified way if one or more material properties are of a random or incompletely known nature?

Meakin [8] states that the upright thoraco–lumbar spine resembles a Euler column buckled in the second mode ($n = 2$). The flexed spine resembles an $n = 1$ buckled column if the antagonistic muscles that control movement increase the bending stiffness (the muscle can act like a variable accidental eccentricity, especially if trauma is present).

Kuchini et al. [9] investigated the formation of subgrain deformation structures arising from the constraining effect of grain

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boundaries on the elastic and plastic deformation fields in polycrystals. The computational approach is based on a Lagragian large-deformation finite-element formulation of the continuum three-dimensional problem. The only discontinuity taken into account at grain boundaries is the change in the orientation of the crystal.

Schafer [10] proposes a new design method that explicitly incorporates local, distortional, and Euler buckling. Elastic buckling analysis, of open cross-section thin-walled columns, typically reveal at least three buckling modes: local, distortional, and Euler (global) (see Schafer [10] for work done in this area in the last two decades). Consistent integration of local, distortional, and Euler buckling into the design of thin-walled columns is needed. Current design specifications (e.g., AISI 1996) give closed-form predictions of the Euler buckling modes.

Dickson [11] determined that a number of factors favor column buckling (Euler's law) and thus the bigger a deformity the more likely it will be to continue progressing, and the taller and more slender the column the more likely it will be to fail and this we see in our patients with idiopathic scoliosis. (the writer notes that this bioengineering interdisciplinary field is progressing rapidly).

Berlin et al. [12] notes that a program was initiated for the possibility of developing active enhancement of structures and, first of all, active stabilization of compressively loaded columns; and to actively control Euler column buckling via electronic means. The buckling control system will also handle the effects of eccentric loading conditions (also accounting for initial eccentricity). The writer feels that such a program, if generally successful, will greatly mitigate Euler buckling if the accidental eccentricities are within the range discussed in this paper.

References [1–4] either deal directly with accidental eccentricity or attempts to determine same. The other references given are the closest this author could find that even hinge on the subject of accidental eccentricity for Euler columns. The author does not believe that much work has been done in this area recently. In any case, the author could find little in the literature that attacks the problem of accidental eccentricity as is done in this paper.

Thus, the objectives of this paper are

1) It simply occurred to the author, one day, to study the buckling of a Euler column if it is compressed at a uniform rate, that is, $y = C_1 t$. Then everywhere y (or Y_{cr}) appears, the time t can be put in its place.

2) To determine a practical range of values of accidental eccentricity e for rectangular sections (see [5,6]). Similar work for thin angles, zees, or channels must deal with local buckling, or crippling, leading to other considerations.

3) To assert, firmly, that P_{cr}/A has virtually no meaning for Euler column buckling, yet is used in virtually every book and paper.

4) To show that the energy required to buckle a Euler column increases as e increases (see Fig. 2).

5) There are virtually no assumptions in this paper, except that Euler's formula, to predict buckling is taken as a proven premise, and D, the moment arm for bending, is assumed as given in Eq. (4).

Preliminaries

Consider testing a pin-ended slender column of length L in compression. The upper platen of the testing machine will be brought down at some uniform rate C_1 until the column fails (buckles). Consider, also, that incipient failure will occur when an element of the column reaches the proportional limit σ_{pl} when the specimen starts to yield if the load is further increased. Again, consider, in practical terms, that no column can be tested that is completely straight, that is, there exists a so-called initial eccentricity e. This eccentricity causes the column to start to bow out just as soon as the load $P > 0$ and continues to bow out further as P increases. In Fig. 1, y represents the shortening of the column which is equal to the total downward movement of the platen. At any x along the column, there is a corresponding bowing out v of the column measured from the initial central undeformed axis of the column. Then the total bending moment at any x, at time t, or corresponding y, is given by

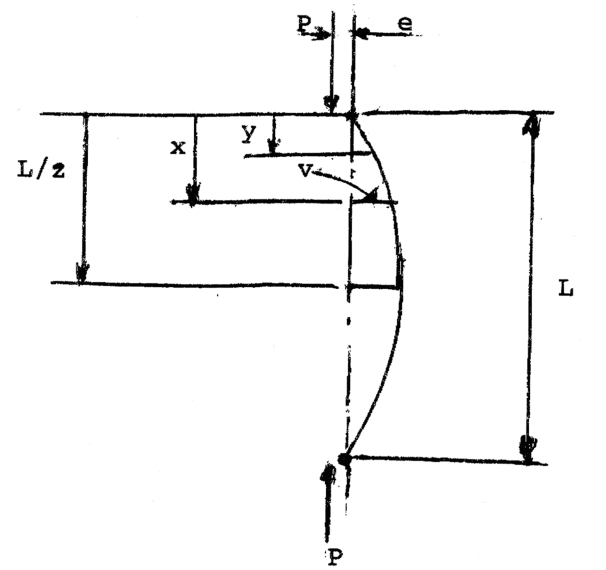

Fig. 1 Column bowed out at some total shortening *y*, and corresponding *P*.

$$M = PD \tag{2}$$

where,

$$D = v + e \tag{3}$$

and

$$D = G(y/y_{cr}) \sin \pi x/L + e \tag{4}$$

Assume a function for D in Eq. (4), where G is simply a multiplier in units of inches, and is further defined via Eq. (21). Now let

$$y = C_1 t, \quad \text{and} \quad y_{cr} = C_1 t_{cr} \tag{5}$$

where the subscripts cr correspond to P_{cr}, that is, the Euler buckling load. Then,

$$y/y_{cr} = t/t_{cr} \tag{6}$$

When $y = y_{cr}$ at $P = P_{cr}$, at $x = L/2$, then

$$D_{cr} = v_{cr} + e \tag{7}$$

To find D_{cr}, determine that fiber that has reached the proportional limit, due to direct compression plus bending at $x = L/2$.

$$\sigma_{pl} = P_{cr}/bd + 6(P_{cr}D_{cr})/bd^2 \tag{8}$$

where b and d are the dimensions of the column, bd is the area, and bending occurs about the weaker axis. Then, from Eq. (8)

$$D_{cr} = (d/6)[(bd\sigma_{pl}/P_{cr}) - 1] \tag{9}$$

Example 1: From Eqs. (1), (2), and (9), find P_{cr}, D_{cr}, and M_{cr}:

Let $b = 1$ in., $d = 3/16$ in., $L = 10$ in., $I = bd^3/12$, $L/r = 184.75$, $E = 3(10)^7$ psi, $\sigma_{pl} = 30,000$ psi, $P_{cr} = 1636.5$ #, $D_{cr} = 0.07616$ in., $M_{cr} = P_{cr}D_{cr} = 1636.5(0.07616) = 124.64$ in#. Thus, the critical moment at $x = L/2$, M_{cr} must not exceed that value so as not to go beyond the proportional limit.

Energy Approach to $P = f(y)$

As was previously stated, e can be an average random eccentricity of the central axis of bending from the resultant crystalline resistance of the steel/carbon matrix, or an initially bent column. The former is still very difficult to determine numerically. An expression can now be written, for P at any time t, expressing the differential change in the external work done by the platen, and to change in the internal stored energy in the column due to direct compression end bending,

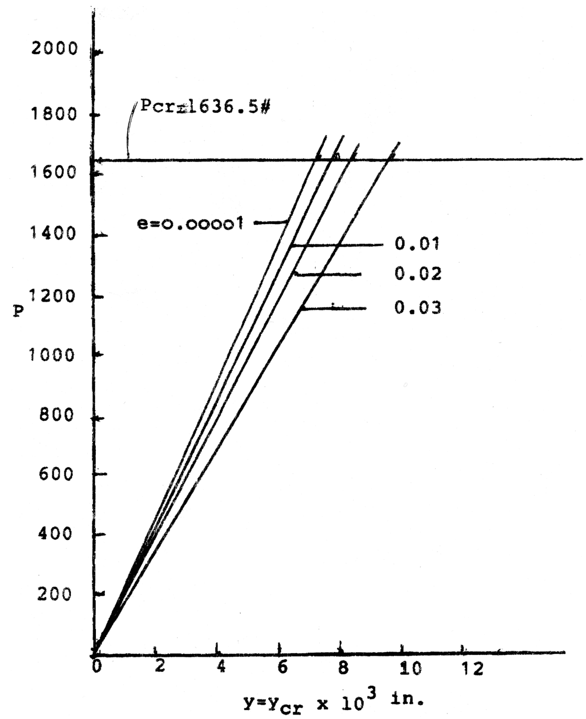

Fig. 2 Plot of P vs y for a set of accidental eccentricities e.

as P goes from P to $P + dP$:

$$(P + dP)dy = [-P^2(L - y) + (P + dP)^2(L - Y)]/AE - \left\{\int (PD)^2 dx - \int [(P + dp)D]^2 dx\right\} \Big/ EI \tag{10}$$

where the limits of integration are from 0 to L. Equation (10) can be shown to reduce to

$$dy/dP = 2(L/AE + e^2 L/EI) + [(8GeL)/(\pi EI y_{cr}) - 2/AE]y + (LG^2/EI)(y/y_{cr})^2 \tag{11}$$

Letting the first term on the RHS of Eq. (11) be $a = 2(L/AE + e^2 L/EI)$, and the second term on the RHS be $b = [(8GeL)/(\pi EI y_{cr}) - 2/AE]$, and the third term on the RHS be $c = L(G/y_{cr})^2/EI$, and $q = 4ac - b^2$ then Eq. (11) can be put in the form of Eq. (12), where Y is the RHS of Eq. (11).

$$P = \int \mathrm{d}y/Y + C \tag{12}$$

The solution of Eqs. (11) and (12) yields

$$P = (2/\sqrt{q})\{\tan^{-1}[(2cy + b)/\sqrt{q}]\} + C \tag{13}$$

Then, Fig. 2 is obtained by first selecting a value of e. Also, for some guessed value of y_{cr}, find a, b, c, q. Then in Eq. (13), solve for C (variable for all guessed values of y_{cr}), by using the initial condition in Eq. (13), $P(0) = 0 = (2/\sqrt{q}[\tan^{-1}(b/\sqrt{q})] + C$. Then, with that value of C, solve Eq. (13) for P, using $y = y_{cr}$ (guessed), and plot it. Repeat procedure for other guessed values until the curve crosses the true P_{cr} ordinate and determine the true y_{cr} abscissa. Then for various values of eccentricity e, from 10^{-5} to $3(10)^{-2}$, Fig. 2 plots P vs y.

It is clear then, as e approaches zero, the external work of the platen applied load is the minimum energy necessary to buckle the slender column. When bending, more energy is needed to reach buckling.

Find Maximum Permissible Value of e and y_{cr}

Solve for maximum v at $x = L/2$. At any time t, the bending moment M is

$$M = PD = P(v + e) = P[(G \sin \pi x/L)(t/t_{cr}) + e] \tag{14}$$

if e where actually zero, then

$$v = (Gt/t_{cr})\{\sin \pi x/L)\tag{15}$$

where Eq. (15) describes a sin curve bowing out with time. Because accidental eccentricity e is never zero, then it is appropriate to look at the interrelationship of a sin curve and a parabola, via D, the moment arm for $M = PD$, where D is given in Eq. (4). To find v and v_{cr} with accidental eccentricity e, place a 1# load acting normal to the column at $x = L/2$, and then $m = x/2$. Then find v from

$$v = \int Mm\,\mathrm{d}x/EI \tag{16}$$

and,

$$v = (2P/EI)\int [(G \sin \pi x/L)(t/t_{cr}) + e](x/2)\,\mathrm{d}x \tag{17}$$

Solving Eq. (17),

$$v = (P/EI)[G(L/\pi)^2(t/t_{cr}) + eL^2/8] \tag{18}$$

and at $x = L/2$ at P_{cr}, maximum v is

$$v_{cr} = (P_{cr}/EI)[G(L/\pi)^2 + eL^2/8] \tag{19}$$

Example 2: Using the data and results from Example 1, the maximum permissible values for G and e can be obtained for $P_{cr} = 1636.5\#$. Using Eqs. (7–9) and (19), first getting v_{cr} and D_{cr}

$$v_{cr} = [1636.5/(3 \times 10^7 \times 5.493 \times 10^{-4})](10.132G + 12.5e) \tag{20}$$

Then,

$$v_{cr} + e = 0.07616 = (1.0062G + 1.241e) + e \tag{21}$$

Let $e \to 0$: then from Eq. (21), max $G = 0.07569$ in.
Let $G \to 0$, then from Eq. (21), max $e \approx 0.03398$ in.

If the accidental eccentricity e is in the neighborhood of 0.03, that is, a considerable percentage of $d/2$, and because bending and bowing out of the column must begin as soon as $P > 0$, then G must reach some value greater than zero when σ_{pl}, the proportional limit, is reached and buckling impends at something less than Euler P_{cr}.

Example 3: Let $e = 0.01$, then $e/(d/2) = 0.1067$. Using the data in example 1,

$$D_{cr} = v_{cr} + e = 0.07616 = (1,0062G + 2,241e) = (1.0062G + 0.02241)$$

$$G = 0.05375 \text{ in.} \quad \text{and} \quad v_{cr} = 0.06616 \text{ in.}$$

Thus, for a value of $e = 0.01$, the column will bow out at or near $v = 0.0538$ in., and σ_{pl} is reached. This bowing out is clearly a combination sin/parabola curve.

And so, in Fig. 2, for values of e equal to 0.00001, 0.01, 0.02, and, 0.03, the corresponding values of y_{cr} are 0.0074, 0.00795, 0.0086, and 0.00975 in., respectively. And so in Eq. (21) as e approaches zero, G approaches v_{cr}. Then, for $e \neq 0$, G is related to v via Eqs. (3) and (4), and so the bowing out v_{cr} due to beam–column interaction, contains a function of e.

Conclusion

This paper makes no attempt to introduce a new method for design of Euler columns. Rather, first of all, it points out the danger of considering the so-called P_{cr}/A as some fictitious low stress as L/r proceeds to get larger and larger, and P_{cr} is approached. This paper gives equations that predict the calculated shape of the bowing out of the column up to and including critical failure, and predicts the time of failure t_{cr}. It also shows that more energy is needed to buckle the Euler column as the accidental eccentricity e increases, and y_{cr}

increases in Fig. 2. In addition, initial accidental eccentricity can be learned via increased metallurgical studies of the random arrangement of the crystalline structure of the metallic grains, in order to determine the probable center of internal resistance of the column (as noted in the references). As the eccentricity $e \to 0$, then the bowing out may start very slowly as $P \to P_{\text{cr}}$, and suddenly build up to v_{cr} and snap out and fail as σ_{pl} is reached. Also, Eq. (21) shows that $P = f(y)$ is not a straight line if the column is compressed at a uniform rate. It was shown that the complete range of values of $0 < e \leq 0.03+$ was noted and graphed.

Recent papers, though few, have not developed the subject as given in this paper.

Acknowledgements

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